# Sparse Domination for Paraproducts – Bilinear Form Version

→ Instead of dominating $|\Pi_{b,Q_0} f(x)|$ pointwise on $Q_0$, work directly with the bilinear form:

$$(\Pi_{b,Q_0} f_1, f_2) = \sum_{Q \subset Q_0} (b, h_Q) \langle f_1 \rangle_Q (f_2, h_Q).$$

We will need certain properties of the localized CZ-decomposition of a function:

## §. Localized CZ-decomposition (Short version)

→ Let $Q_0 \in \mathcal{D}$, $f \in L^1(Q_0)$, $f \neq 0$, and $\alpha > 1$. Consider the collection:

$$\mathcal{E} := \{\text{maximal subcubes } R \subset Q_0 \text{ s.t. } \langle |f| \rangle_R > \alpha \langle |f| \rangle_{Q_0}\} \quad \& \quad E := \bigcup_{R \in \mathcal{E}} R$$

The immediate usual properties:

- $\alpha \langle |f| \rangle_{Q_0} < \langle |f| \rangle_R \leq 2^n \alpha \langle |f| \rangle_{Q_0} \ \forall R \in \mathcal{E}$
- $\boxed{\sum_{R \in \mathcal{E}} |R| \leq \frac{1}{\alpha} |Q_0|}$ SPARSENESS
- $f(x) \leq \alpha \langle |f| \rangle_{Q_0}$ a.e. $x \in Q_0 \setminus E$ (LDT).

$$R \in \mathcal{E} \Rightarrow |R| < \frac{1}{\alpha \langle |f| \rangle_{Q_0}} \int_R |f|$$

$$\Rightarrow \sum_{R \in \mathcal{E}} |R| < \frac{|Q_0|}{\alpha \int_{Q_0} |f|} \underbrace{\sum_{R \in \mathcal{E}} \int_R |f|}_{\leq \int_{Q_0} |f|} \leq \frac{1}{\alpha} |Q_0|.$$

→ Define then the "good function"

$$g(x) := f(x) - \sum_{R \in \mathcal{E}} (f(x) - \langle f \rangle_R) \mathbb{1}_R(x) = \begin{cases} f(x), & x \in Q_0 \setminus E \\ \langle f \rangle_R, & x \in R, R \in \mathcal{E}. \end{cases}$$

We need the following properties of $g$:

→ $\int_{Q_0} |g| \leq \int_{Q_0 \setminus E} |f| + \sum_{R \in \mathcal{E}} \int_R |\langle f \rangle_R| \leq \int_{Q_0 \setminus E} |f| + \sum_{R \in \mathcal{E}} \int_R |f| = \int_{Q_0} |f|$ $\quad \boxed{\int_{Q_0} |g| \leq \int_{Q_0} |f|}$

→ $x \in Q_0 \setminus E \Rightarrow |g(x)| = |f(x)| \leq \alpha \langle |f| \rangle_{Q_0}$ a.e.
$x \in R, R \in \mathcal{E} \Rightarrow |g(x)| = |\langle f \rangle_R| \leq 2^n \alpha \langle |f| \rangle_{Q_0}$
$\Rightarrow \boxed{|g(x)| \leq 2^n \alpha \langle |f| \rangle_{Q_0} \text{ a.e. on } Q_0}$

$$\Rightarrow \left(\int_{Q_0} |g|^2\right)^{1/2} \leq \left(2^n \alpha \langle |f| \rangle_{Q_0} \int_{Q_0} |g|\right)^{1/2} \leq \left(2^n \alpha |Q_0| \langle |f| \rangle_{Q_0}^2\right)^{1/2} \quad \boxed{\|g\|_{L^2} \leq (2^n \alpha |Q_0|)^{1/2} \langle |f| \rangle_{Q_0}}$$

→ Suppose $Q \subset Q_0$, but $Q \not\subset E$.

$$(f, h_Q) = (g, h_Q) + \underbrace{\sum_{R \in \mathcal{E}} ((f - \langle f \rangle_R) \mathbb{1}_R, h_Q)}$$

$$= \sum_{\substack{R \in \mathcal{E} \\ R \subsetneq Q}} ((f - \langle f \rangle_R) \mathbb{1}_R, h_Q) = \sum_{\substack{R \in \mathcal{E} \\ R \subsetneq Q}} \underbrace{\left(\int_R (f - \langle f \rangle_R)\right)}_{0} h_Q(R) = \boxed{0}$$

$$\boxed{(f, h_Q) = (g, h_Q) \ \forall Q \not\subset E}$$

Remark: Similarly, $\langle f \rangle_Q = \langle g \rangle_Q, \ \forall Q \not\subset E$

$$\int_Q f\,dx = \int_Q g\,dx + \underbrace{\sum_{\substack{R \in \mathcal{E} \\ R \subsetneq Q}} \int_R (f - \langle f \rangle_R)}_{0}$$

⇒ On all $Q \subset Q_0$, $Q \not\subset E$, the Haar coefficients & the averages of $f$ can be replaced by those for $g$!

## Bilinear Form Sparse Domination for $\Pi_b$:

→ Recall: by John–Nirenberg & passing to the square function:
$$\|b\|_{BMO} \simeq \sup_Q \left(\frac{1}{|Q|}\sum_{P\subset Q}(b,h_P)^2\right)^{1/2}.$$

→ Work within a fixed $Q_0 \in \mathcal{D}$, and look at:
$$(\Pi_{b,Q_0} f_1, f_2) = \sum_{Q\subset Q_0}(b,h_Q)\langle f_1\rangle_Q (f_2,h_Q).$$

Let $\alpha > 1$ and for $i=1,2$ form the collections: $\mathcal{E}_i := \{\text{maximal } R\subset Q_0 \text{ s.t. } \langle |f_i|\rangle_R > 2\alpha \langle |f_i|\rangle_{Q_0}\}$ and put $E_i := \bigcup_{R\in\mathcal{E}_i} R$. Also let the "good" functions:
$$g_i := f_i - \sum_{R\in\mathcal{E}_i}(f_i - \langle f_i\rangle_R)\mathbb{1}_R$$
$$|E_i| < \frac{1}{2\alpha}|Q_0|$$

Finally, let $\mathcal{E} := \{\text{maximal } R\in\mathcal{D}(Q_0) \text{ contained in } E_1\cup E_2\}$ and $E := \bigcup_{R\in\mathcal{E}} R$.
$$\Rightarrow \sum_{R\in\mathcal{E}}|R| \le \sum_{R\in\mathcal{E}_1}|R| + \sum_{R\in\mathcal{E}_2}|R| < 2\cdot\frac{1}{2\alpha}|Q_0| \Rightarrow \boxed{\sum_{R\in\mathcal{E}}|R| < \frac{1}{\alpha}|Q_0|}$$

Separate the $Q\subset Q_0$ into two categories:
$$\left|(\Pi_{b,Q_0}f_1,f_2)\right| = \left|\sum_{Q\subset Q_0}(b,h_Q)\langle f_1\rangle_Q(f_2,h_Q)\right|$$
$$\le \underbrace{\sum_{\substack{Q\subset Q_0\\ Q\not\subset E}}|(b,h_Q)|\langle |f_1|\rangle_Q|(f_2,h_Q)|}_{} + \underbrace{\sum_{R\in\mathcal{E}}\left|\sum_{P\subset R}(b,h_P)\langle f_1\rangle_P(f_2,h_P)\right|}_{\text{will recurse on these}}$$

$$Q\not\subset E \Rightarrow Q\not\subset E_1 \Rightarrow \langle |f_1|\rangle_Q \le 2\alpha\langle |f_1|\rangle_{Q_0}$$
$$\Rightarrow Q\not\subset E_2 \Rightarrow (f_2,h_Q) = (g_2,h_Q)$$

$$\Rightarrow \sum_{\substack{Q\subset Q_0\\ Q\not\subset E}}|(b,h_Q)|\langle |f_1|\rangle_Q|(f_2,h_Q)| \le 2\alpha\langle |f_1|\rangle_{Q_0}\sum_{\substack{Q\subset Q_0\\ Q\not\subset E}}|(b,h_Q)||(g_2,h_Q)|$$
$$\le 2\alpha\langle |f_1|\rangle_{Q_0}\underbrace{\left(\sum_{Q\subset Q_0}(b,h_Q)^2\right)^{1/2}}_{\le\sqrt{|Q_0|}\,\|b\|_{BMO}}\underbrace{\left(\sum_{Q\subset Q_0}(g_2,h_Q)^2\right)^{1/2}}_{\le \|g_2\|_{L^2}\le\sqrt{2^n\alpha|Q_0|}\,\langle |f_2|\rangle_{Q_0}}$$
$$\lesssim \|b\|_{BMO}\langle |f_1|\rangle_{Q_0}\langle |f_2|\rangle_{Q_0}|Q_0|$$

$$\Rightarrow |(\Pi_{b,Q_0}f_1,f_2)| \lesssim \|b\|_{BMO}\langle |f_1|\rangle_{Q_0}\langle |f_2|\rangle_{Q_0}|Q_0| + \sum_{R\in\mathcal{E}}\left|\sum_{P\subset R}(b,h_P)\langle f_1\rangle_P(f_2,h_P)\right|$$
$$\sum_{R\in\mathcal{E}}|(\Pi_{b,R}f_1,f_2)|$$

The collection $\mathcal{E}$ becomes the $\mathcal{S}$-children of $Q_0$, then we recurse on the sums over $R\in\mathcal{E}$
$$\Rightarrow |(\Pi_{b,Q_0}f_1,f_2)| \lesssim \|b\|_{BMO}\sum_{Q\in\mathcal{S}(Q_0)}\langle |f_1|\rangle_Q\langle |f_2|\rangle_Q|Q|$$

$$\Rightarrow \boxed{|(\Pi_{b,Q_0}f_1,f_2)| \lesssim \|b\|_{BMO}\left(\mathcal{A}_{\mathcal{S}(Q_0)}|f_1|, |f_2|\right)}$$

Take again $b$ w/ finite Haar expansion, $f_1\in L^2(w)$, $f_2\in L^2(w^{-1})$ ⇒ linear $A_2$ bound.

# Localized CZ decomposition

→ frequently applied to a compactly supported function in the construction of sparse collections.

Let $f\in L^1(Q_0)$, $f\neq 0$ (i.e. $\operatorname{supp}(f)\subset Q_0$ and $0<\int_{Q_0}|f|\,dx<\infty$), where $Q_0\in\mathcal{D}$ is a fixed dyadic cube.
For some $\alpha>1$, consider the collection:

$$\boxed{\mathcal{E}:=\{\text{maximal subcubes } R\in\mathcal{D}(Q_0) \text{ s.t. } \langle|f|\rangle_R>\alpha\langle|f|\rangle_{Q_0}\}} \qquad \boxed{E:=\bigcup_{R\in\mathcal{E}}R}$$

Some immediate properties:

→ $\boxed{\alpha\langle|f|\rangle_{Q_0}<\langle|f|\rangle_R\le 2^n\alpha\langle|f|\rangle_{Q_0},\ \forall R\in\mathcal{E}}$ $R\in\mathcal{E}\Rightarrow$ dyadic parent $\hat{R}$ of $R$ was not selected $\Rightarrow\langle|f|\rangle_{\hat{R}}\le\alpha\langle|f|\rangle_{Q_0}$ $\Rightarrow\langle|f|\rangle_R=\frac{1}{|R|}\int_R|f|\le\frac{2^n}{|\hat{R}|}\int_{\hat{R}}|f|\le 2^n\alpha\langle|f|\rangle_{Q_0}$.

→ $\boxed{\sum_{R\in\mathcal{E}}|R|\le\frac{1}{\alpha}|Q_0|}$ $R\in\mathcal{E}\Rightarrow|R|<\frac{1}{\alpha\langle|f|\rangle_{Q_0}}\int_R|f|\Rightarrow\sum_{R\in\mathcal{E}}|R|\le\frac{1}{\alpha\langle|f|\rangle_{Q_0}}\int_E|f|\le\frac{1}{\alpha\langle|f|\rangle_{Q_0}}\int_{Q_0}|f|=\frac{1}{\alpha}|Q_0|$.

→ $\boxed{f(x)\le\alpha\langle|f|\rangle_{Q_0},\ \text{a.e. } x\in Q_0\setminus E}$ (LDT).

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Now define the bad functions:

$$\boxed{b_R(x):=(f(x)-\langle f\rangle_R)\mathbb{1}_R(x),\ \forall R\in\mathcal{E}}$$

- → $\operatorname{supp}(b_R)\subset R$ & $\int_R b_R\,dx=0$, $\forall R\in\mathcal{E}$
- → $\langle b_R\rangle_Q=\langle f\rangle_Q-\langle f\rangle_R$, $\forall R\in\mathcal{E}$, $Q\subset R$
- → $(f,h_Q^\varepsilon)=(b_R,h_Q^\varepsilon)$, $\forall R\in\mathcal{E}$, $Q\subset R$.

→ $\boxed{\int_R|b_R|\,dx\le 2^{n+1}\alpha|R|\langle|f|\rangle_{Q_0},\ \forall R\in\mathcal{E}}$

$$\int_R|b_R|\,dx=\int_R|f(x)-\langle f\rangle_R|\,dx \le 2|R|\langle|f|\rangle_R\le 2^{n+1}\alpha|R|\langle|f|\rangle_{Q_0}$$

→ $\boxed{\|b\|_1=\sum_{R\in\mathcal{E}}\int_R|b_R|\,dx\le 2^{n+1}\int_{Q_0}|f|=2^{n+1}\|f\|_1.}$

$$\sum_{R\in\mathcal{E}}\int_R|b_R|\,dx\le 2^{n+1}\alpha\langle|f|\rangle_{Q_0}\sum_{R\in\mathcal{E}}|R| \le 2^{n+1}\alpha\int_{Q_0}|f|=2^{n+1}\|f\|_1.$$

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Take some $Q\not\subset E$?

$$\left(\int_Q|g|^p\right)^{1/p}\le\left(\int_Q|g|\right)^{1/p}\|g\|_\infty^{\frac{p-1}{p}} \qquad \tfrac{p-1}{p}=1-\tfrac{1}{p}=\tfrac{1}{p'}$$

$$\le\left(\int_Q|f|\right)^{1/p}(2^n\alpha)^{1/p'}\langle|f|\rangle_{Q_0}^{1/p'}$$

$$\Rightarrow Q\not\subset E:\ \left(\int_Q|g|^p\right)^{1/p}\le\left(\int_Q|f|\right)^{1/p}\left(2^n\alpha\langle|f|\rangle_{Q_0}\right)^{1/p'}$$

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$$Q\not\subset E:\ \int_Q|g|=\int_{Q\setminus E}|g|+\sum_{\substack{R\in\mathcal{E}\\R\subset Q}}\int_R|g|$$
$$=\int_{Q\setminus E}|f|+\sum_{\substack{R\in\mathcal{E}\\R\subset Q}}\int_R|\langle f\rangle_R|$$
$$\le\int_{Q\setminus E}|f|+\sum_{\substack{R\in\mathcal{E}\\R\subset Q}}\int_R\langle|f|\rangle_R$$
$$=\int_{Q\setminus E}|f|+\sum_{\substack{R\in\mathcal{E}\\R\subset Q}}\int_R|f|=\int_Q|f|$$

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and the good function:

$$\boxed{g(x):=f(x)-\sum_{R\in\mathcal{E}}b_R(x)=f(x)-\sum_{R\in\mathcal{E}}(f(x)-\langle f\rangle_R)\mathbb{1}_R(x)}$$

→ $g(x)=\begin{cases}f(x), & x\in Q_0\setminus E\\ \langle f\rangle_R, & x\in R,\ R\in\mathcal{E}\end{cases}$

→ $\int_Q g=\int_Q f$ & $\int_Q|g|\le\int_Q|f|$, $\forall Q\in\mathcal{D}(Q_0)$, $Q\not\subset E$

→ $(g,h_Q^\varepsilon)=(f,h_Q^\varepsilon)$, $\forall Q\in\mathcal{D}(Q_0)$, $Q\not\subset E$.

Remark: since in the construction above, $Q_0\notin\mathcal{E}$, we have in particular that:

→ $\boxed{\int_{Q_0}|g|\le\int_{Q_0}|f| \text{ i.e. } \|g\|_{L^1(Q_0)}\le\|f\|_{L^1(Q_0)}<\infty}$

→ $\boxed{|g(x)|\le 2^n\alpha\langle|f|\rangle_{Q_0} \text{ a.e. } x\in Q_0 \text{ i.e. } \|g\|_{L^\infty(Q_0)}\le 2^n\alpha\langle|f|\rangle_{Q_0}<\infty}$

$x\in Q_0\setminus E\Rightarrow|g(x)|=|f(x)|\le\alpha\langle|f|\rangle_{Q_0}$
$x\in R, R\in\mathcal{E}\Rightarrow|g(x)|=|\langle f\rangle_R|\le\langle|f|\rangle_R\le 2^n\alpha\langle|f|\rangle_{Q_0}$.

→ $\boxed{\underbrace{\left(\int_{Q_0}|g|^p\right)^{1/p}}_{\|g\|_{L^p(Q_0)}}\le(2^n\alpha)^{1/p'}|Q_0|^{1/p}\langle|f|\rangle_{Q_0},\ \forall p>1 \qquad \|g\|_{L^p(Q_0)}\le\left(\frac{2^n\alpha}{|Q_0|}\right)^{1/p'}\int_{Q_0}|f|=\left(\frac{2^n\alpha}{|Q_0|}\right)^{1/p'}\|f\|_{L^1(Q_0)}}$

Take $p=2$: $\|g\|_{L^2(Q_0)}\le\left(\frac{2^n\alpha}{|Q_0|}\right)^{1/2}\int_{Q_0}|f|=(2^n\alpha|Q_0|)^{1/2}\langle|f|\rangle_{Q_0}$

$$\|g\|_{L^p(Q_0)}\le\|g\|_{L^1(Q_0)}^{1/p}\|g\|_{L^\infty(Q_0)}^{1/p'}$$
$$\le\|f\|_{L^1(Q_0)}^{1/p}(2^n\alpha)^{1/p'}\|f\|_{L^1(Q_0)}^{1/p'}\frac{1}{|Q_0|^{1/p'}}$$
$$=\left(\frac{2^n\alpha}{|Q_0|}\right)^{1/p'}\int_{Q_0}|f|=\left(\frac{2^n\alpha}{|Q_0|}\right)^{1/p'}\|f\|_{L^1(Q_0)}.$$

**Remark:** Some features of the underlying structure of this decomposition are universally true: given $Q_0 \in \mathcal{D}$, $f \in L^1(Q_0)$, say $\mathcal{E} \subset \mathcal{D}(Q_0)$ is any collection of <u>pairwise disjoint</u> subcubes of $Q_0$ and $E := \bigcup_{R \in \mathcal{E}} R$. (Typically, $\mathcal{E}$ will consist of the maximal subcubes of $Q_0$ satisfying some "stopping condition", and will be chosen in such a way as to satisfy a sparseness condition $|E| < \frac{1}{\alpha}|Q_0|$, $\alpha > 1$).

Since the elements of $\mathcal{E}$ are "bad" cubes in some sense, we let for every $R \in \mathcal{E}$:

$$b_R(x) := (f(x) - \langle f \rangle_R) \mathbb{1}_R(x), \quad \forall R \in \mathcal{E},$$

the "bad" functions. These do have some nice properties: $\operatorname{supp}(b_R) \subset R$, $\forall R \in \mathcal{E}$ and $\int_R b_R \, dx = 0$, $\forall R \in \mathcal{E}$. Finally, let

$$g(x) := f(x) - \sum_{R \in \mathcal{E}} b_R(x) = \begin{cases} f(x), & x \in Q_0 \setminus E \\ \langle f \rangle_R, & x \in R, R \in \mathcal{E}. \end{cases}$$

Note that $g$ will be just $f$ outside of $E$, and is constant on each "bad" cube. The function $g$ will be the "good" function – in the CZ decomposition of $f$, $g$ will be in both $L^1(Q_0)$ and $L^\infty(Q_0)$, therefore in all $L^p(Q_0)$. Moreover, the $L^p$-norms of $g$ can be bounded above by the $L^1$-norm of $f$.

Crucial property of any such construction: for any $Q \in \mathcal{D}(Q_0)$, $Q \not\subset E$, the following hold:

$$\boxed{\int_Q f \, dx = \int_Q g \, dx} \quad \& \quad \boxed{(f, h_Q^\varepsilon) = (g, h_Q^\varepsilon)} \quad \& \quad \boxed{\int_Q |g| \, dx \leq \int_Q |f| \, dx}$$

In other words: whenever we consider the <u>average</u> of $f$ or a <u>Haar coefficient</u> of $f$ <u>over a cube not contained in $E$</u>, we can simply work with the "good" function instead.

<u>**NOTE**</u>: These observations are true regardless of $f$ having a connection with $\mathcal{E}$ or not.

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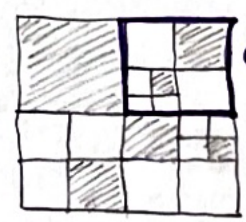
$Q \in \mathcal{D}(Q_0)$, $Q \not\subset E$:



$Q \not\subset E$

- $$\int_Q f \, dx = \int_Q \Big(g + \sum_{R \in \mathcal{E}} b_R\Big) dx = \int_Q g \, dx + \sum_{\substack{R \in \mathcal{E} \\ R \subset Q}} \underbrace{\int_R b_R \, dx}_{0} = \int_Q g \, dx.$$
- $$(f, h_Q^\varepsilon) = (g, h_Q^\varepsilon) + \sum_{\substack{R \in \mathcal{E} \\ R \subset Q}} \underbrace{(b_R, h_Q^\varepsilon)}_{0 \text{ b/c } h_Q^\varepsilon \text{ is constant on } R} = (g, h_Q^\varepsilon)$$
- $$\int_Q |g| \, dx = \int_{Q \setminus E} |f| \, dx + \sum_{\substack{R \in \mathcal{E} \\ R \subset Q}} \underbrace{\int_R |\langle f \rangle_R| \, dx}_{\leq |R| \langle |f| \rangle_R = \int_R |f|} \leq \int_{Q \setminus E} |f| \, dx + \underbrace{\sum_{\substack{R \in \mathcal{E} \\ R \subset Q}} \int_R |f|}_{= \int_{Q \cap E} |f|} = \int_Q |f| \, dx.$$